Model Answer

AV-8871

B.Sc. (Hon's) (First Semester) Exam. - 2015-16

Mathematics (Analytical Geometry of Three-Dimensions)

Director circle:-

1 (i) Locus of the point of intersection of two perpendicular tangents is called Director circle.
Equation of director circle for the conic $\frac{l}{r} = 1 + e\cos\theta$

is $r^2(1-e^2) + 2elr\cos\theta - 2l^2 = 0$.

(ii) Given $\frac{c}{r} = A\cos\theta + B\sin\theta$

change in cartesian form

$$c = Ar\cos\theta + Br\sin\theta$$

$$= Ax + By \quad \text{where } x = r\cos\theta,\ y = r\sin\theta$$

which is also represent a straight line

(iii) Radical centre - The four radical lines of 4 spheres taken three at a time intersect at a point is called radical centre of 4 spheres.

If $S_1 = 0, S_2 = 0, S_3 = 0, S_4 = 0$

then $S_1 = S_2 = S_3$, $S_1 = S_2 = S_4$, $S_2 = S_3 = S_4$, $S_1 = S_3 = S_4$

these radical lines passes through the pt. $S_1 = S_2 = S_3 = S_4$ which is known as radical centre.

(iv) Limiting point of co-axial system:- Centre of spheres of zero radii of a co-axial system of spheres are called limiting point of system.

P.T.O.

(v) <u>Cone:-</u> A cone is a surface generated by a straight line which passes through a fixed point and satisfies one more condition, as it may intersect a given curve or touch a given surface.

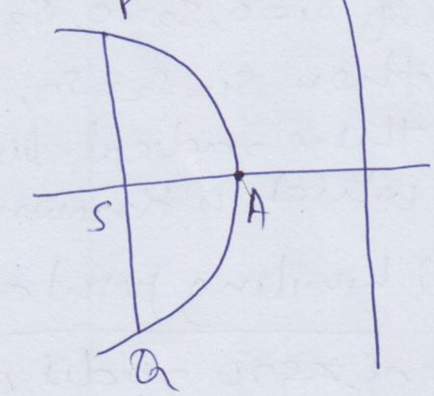
(vi) <u>Right circular cylinder:-</u> A right circular cylinder is a surface generated by a line which intersect a fixed circle, called the guiding circle, and is ⊥er to its plane.

(vii) <u>Director sphere for a conicoid:-</u> Director sphere of central conicoid is locus of point of intersection of three mutually ⊥er tangent planes to C.C.

Eqn. of D.S. is $x^2+y^2+z^2 = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

(viii) If $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ be two planes and θ be the angle between them.

then $\cos\theta = \dfrac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}}$



2. We have to prove that

$SQ = \left(\dfrac{1+3e^2+e^4}{1+e^2-e^4}\right)\cdot l$

Now, eqn. of conic is

$\frac{l}{r} = 1+e\cos\theta \longrightarrow (1)$

The co-ordinate of one extremity P is $(l, \pi/2)$

Then eqn. of normal at $(l, \pi/2)$ is:

$$\frac{l}{r}\left[\frac{e\sin\pi/2}{1+e\cos\pi/2}\right] = e\sin\theta + \sin(\theta-\pi/2)$$

$$\frac{le}{r} = e\sin\theta - \cos\theta \Rightarrow \frac{le}{r}+\cos\theta = e\sin\theta \longrightarrow (2)$$

from (1), $\cos\theta = \frac{1}{e}\left(\frac{l}{r}-1\right) = \left(\frac{l-r}{er}\right)$

from (1) and (2), we get.

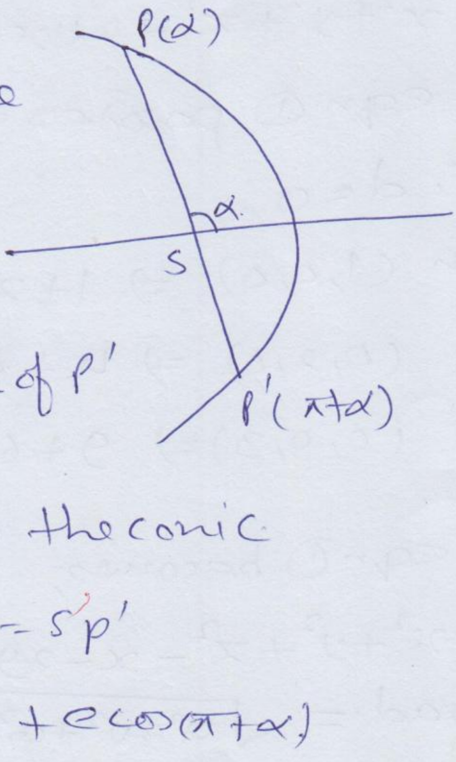
$$\frac{le}{r} + \left(\frac{l-r}{er}\right) = e\sqrt{1-\left(\frac{l-r}{er}\right)^2}.$$

$$le^2 + (l-r) = e\sqrt{e^2r^2-(l-r)^2}$$

squaring both side and solve.

$$r = \left(\frac{1+3e^2+e^4}{1+e^2-e^4}\right)\cdot l$$

3. Let the eqn of the conic be



$\frac{l}{r} = 1+e\cos\theta$. Also, let

PSP' be any focal chord of this conic and vectorial angle of P' is $\pi+\alpha$.

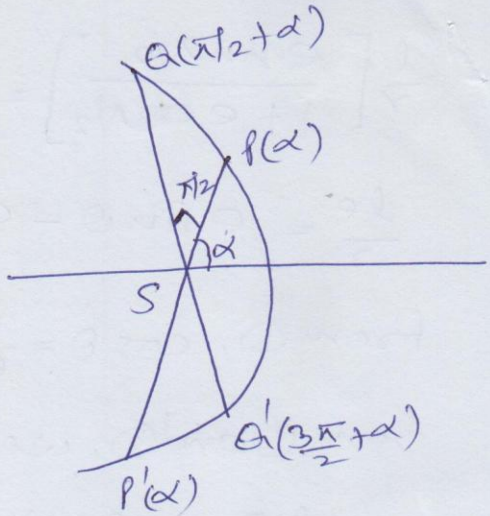
(a) Since $P(\alpha)$ and $P'(\pi+\alpha)$ lie on the conic

$\therefore \theta=\alpha,\ r=SP,\ \theta=\pi+\alpha,\ r=SP'$

$$\frac{l}{SP} = 1+e\cos\alpha,\quad \frac{l}{SP'} = 1+e\cos(\pi+\alpha)$$

$$\frac{1}{SP}+\frac{1}{SP'} = 2/l = \text{const}.$$

P.T.O.



(b). $\because$ Q and Q' lie on the conic

$\therefore \frac{l}{SQ} = 1+e\cos(\pi/2+\alpha)$

and $\frac{l}{SQ'} = 1+e\cos(\frac{3\pi}{2}+\alpha)$

from Part(a) $PP' = \frac{2l}{1-e^2\cos^2\alpha}$

$QQ' = \frac{2l}{1-e^2\sin^2\alpha}$

$\Rightarrow \frac{1}{PP'} + \frac{1}{QQ'} = \frac{2-e^2}{2l}$ constt.

4. Let the eqn. of sphere be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \longrightarrow (1)$

$\because$ eqn (1) passes through the origin

$\therefore d=0$

Again $(1,0,0) \Rightarrow 1+2u = 0 \Rightarrow u=-1/2$

$(0,2,0) \Rightarrow 4+4v=0 \Rightarrow v=-1$

$(0,0,3) \Rightarrow 9+6w=0 \Rightarrow w=-3/2$

$\therefore$ eqn (1) becomes,

$x^2+y^2+z^2-x-2y-3z=0$

rad. $= \sqrt{u^2+v^2+w^2-d} = \sqrt{\frac{1}{4}+1+\frac{9}{4}} = \frac{\sqrt{14}}{2}$

centre $= (-u,-v,-w) = (\frac{1}{2}, 1, \frac{3}{2})$.

P.T.O.

5(a) Let the eqn. of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

Since $y=mx, z=c$ meets it, where

$$(1+m^2)x^2+2(u+vm)x+(c^2+2wc+d)=0$$

the values of x must be coincident.

$$(u+vm)^2=(1+m^2)\cdot(c^2+2wc+d) \longrightarrow ①$$

Similarly

$$(u-vm)^2=(1+m^2)(c^2-2wc+d) \longrightarrow ②$$

Subtracting ② from ①, we get

$$4uvm = 4wz(1+m^2)$$

Hence the locus of $(-u,-v,-w)$ be

$$xym+zc(1+m^2)=0.$$

(b). Any plane through (a,b,c) is

$$l(x-a)+m(y-b)+n(z-c)=0 \longrightarrow ①$$

and the line ⊥er to it from the origin is

$$\frac{x}{l}=y/m=z/n. \longrightarrow ②$$

The foot of the ⊥er is the point of intn. of ① and ②

from ① and ②

$$x(x-a)+y(y-b)+z(z-c)=0$$

$$\Rightarrow x^2+y^2+z^2-ax-by-cz=0.$$

6. (a) We know that the general equation of cone through the three axis is

$$fyz+gzx+hxy=0 \longrightarrow ①$$

Its reciprocal cone is

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0 \longrightarrow ②$$

where $A = bc - f^2 = -f^2, B = ac - g^2 = -g^2, C = ab - h^2 = -h^2$

$F = gh - af = gh, G = hf - bg = hf, H = fg - ch = fg.$

Putting above value in (2), we get.

$-f^2x^2 - g^2y^2 - h^2z^2 + 2ghyz + 2hfzx + 2fgxy$

$\Rightarrow (fx + gy - hz)^2 = 4fgxy$

$fx + gy - hz = \pm 2\sqrt{fgxy}$

$\Rightarrow (\sqrt{fx} \pm \sqrt{gy})^2 = hz$

$\Rightarrow \sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$

6 (b) Given cone is $ax^2 + by^2 + cz^2 = 0$ whose reciprocal $\longrightarrow (1)$

cone is

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0 \longrightarrow (2)$$

from (1) $f = g = h = 0$

from (1) $A = bc - f^2 = bc, B = ac - g^2 = ac, C = ab - h^2 = ab$

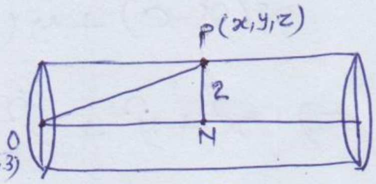
$F = 0 = G = H$

$\therefore$ eqn (2) becomes

$bcx^2 + acy^2 + abz^2 = 0$

$\Rightarrow \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0.$

7.



Radius of Right circular cylinder

PN = 2

P(x, y, z) is any point on Right circular Cylinder.

Axis of cylinder $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{2}$ is ON.

O(1, 2, 3) is point on Axis.

D.C. of Axis is $\frac{2}{\sqrt{4+1+4}}, \frac{1}{\sqrt{4+1+4}}, \frac{2}{\sqrt{4+1+4}}$ ie $\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$

ON = Projection of OP on Axis ON

$= (x-1)\frac{2}{3} + (y-2)\frac{1}{3} + (z-3).\frac{2}{3}$

ΔPON $\quad OP^2 = ON^2 + PN^2$

$$(x-1)^2+(y-2)^2+(z-3)^2 = \left\{\frac{2(x-1)}{3} + \frac{(y-2)}{3} + \frac{2(z-3)}{3}\right\}^2 + 4$$

$$(x-1)^2+(y-2)^2+(z-3)^2 = \frac{1}{9}\{2x-2+y-2+2z-6\}^2+4$$

$$x^2+1-2x+y^2+4-4y+z^2+9-6z = \frac{1}{9}\{2x+y+2z-10\}^2+4$$

$$9(x^2+y^2+z^2-2x-4y-6z+14) = (2x+y)^2+(2z-10)^2+2(2x+y)(2z-10)+36$$

$$9x^2+9y^2+9z^2-18x-36y-54z+126 = 4x^2+y^2+4xy+4z^2+100-40z$$
$$+8zx-40x+4yz-20y+36$$

$$5x^2+8y^2+5z^2-4xy-4yz-8zx+22x-16y-14z-10=0$$

8. Tangent plane to the central conicoid $ax^2+by^2+cz^2=1$ is

$$lx+my+nz = \sqrt{l^2/a+m^2/b+n^2/c}$$

Change in intercept form. Intercept with co-ordinate axis

$$P = \left(\frac{1}{l}\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}, 0, 0\right)$$

$$Q = \left(0, \frac{1}{m}\sqrt{l^2/a+\frac{m^2}{b}+\frac{n^2}{c}}, 0\right)$$

$$R = \left(0, 0, \frac{1}{n}\sqrt{l^2/a+m^2/b+n^2/c}\right)$$

If (x_1, y_1, z_1) be the centroid of ΔPQR, then

$$x_1 = \frac{1}{3l}\sqrt{l^2/a+m^2/b+\frac{n^2}{c}},\ y_1 = \frac{1}{3m}\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+n^2/c},\ z_1 = \frac{1}{3n}\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}$$

$$\Rightarrow \frac{9l^2}{a}+\frac{9m^2}{b}+\frac{9n^2}{c} = \left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)\left(\frac{1}{ax_1^2}+\frac{1}{by_1^2}+\frac{1}{cz_1^2}\right)$$

$$\Rightarrow \frac{1}{ax_1^2}+\frac{1}{by_1^2}+\frac{1}{cz_1^2} = 9.$$

Hence Required locus is $\frac{1}{ax^2}+\frac{1}{by^2}+\frac{1}{cz^2} = 9.$

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